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The vacuum Einstein equations via holonomy around closed loops on characteristic surfaces

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Abstract

We reformulate the standard local equations of general relativity for asymptotically flat space–times in terms of two non-local quantities, the holonomy H around certain closed null loops on characteristic surfaces and the light cone cut function Z, which describes the intersection of the future null cones from arbitrary space–time points, with future null infinity.

We obtain a set of differential equations for H and Z equivalent to the vacuum Einstein equations. By finding an algebraic relation between H and Z and integrating a linear o.d.e. these equations are reduced to just two coupled equations: an integro-differential equation for Z which yields the conformal structure of the underlying space–time and a linear differential equation for the "vacuum" conformal factor. These equations, which apply to all vacuum asymptotically flat space–times are however lengthy and complicated. They nevertheless are amenable to an attractive perturbative scheme which has Minkowski space as a zeroth order solution.

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1. Introduction

The vacuum Maxwell fields on Minkowski space can be expressed as Kirchhoff integrals taken over initial data, i.e., data given on a Cauchy surface. If we replace the spacelike surface by a characteristic, or null, surface, we get the analog to the Kirchhoff integral, the D'Adhemar integral [1]. Maxwell and Yang–Mills fields on both flat and asymptotically

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flat space–times have been given such D'Adhemar-like reformulations with considerable success [2–4]. The present work is an attempt to extend this technique to general relativity. We derive differential equations whose solutions would allow us to express the gravitational field at any given space–time point as a D'Adhemar-like integral over free data given on a characteristic surface.

First consider the well-understood analog to our problem, determining the Maxwell field at some interior point on Minkowski space. Free data is given on the null surface $\mathcal{I}$, which is the three-dimensional boundary of the space–time manifold $\mathcal{M}$. The future light cone emanating from a given interior point x^a would intersect the future half $\mathcal{I}^+$ of $\mathcal{I}$, the intersection being a 2-surface. We refer to this 2-surface as the "light cone cut" of $\mathcal{I}^+$ associated with x^a. The D'Adhemar formulation of the Maxwell equations involves integrals over the free data given on the light cone cut. It is crucial that the light cone structure of the underlying manifold be known, in order to perform this integration. Although for the Minkowski metric this is readily available, it takes considerable effort to obtain this information for an *arbitrary* background. A generalization of the D'Adhemar formulation of the Maxwell field in Minkowski space to the case when the background is curved is available [4], although the resulting equations become complicated. A further generalization of this technique to Yang–Mills fields on both flat and asymptotically flat space–times are also available [5]. In these (Yang–Mills) cases, in addition to the above mentioned difficulty in the determination of the light cones, the final equations are further complicated by the introduction of non-linearity. The D'Adhemar integral (for the Yang–Mills case and the Maxwell case on a curved background) shows explicitly that propagation becomes non-Huygens—i.e., propagation is not confined to the light cone.

A D'Adhemar-like formulation for general relativity might seem inaccessible since the gravitational field is not separable from the background and its characteristic surfaces: it requires *simultaneously* solving for the field and the null surfaces. It is nevertheless possible to give such a formulation by dividing the equations into two parts: one resembling the D'Adhemar integrals for a (self-dual or anti-self-dual) Yang–Mills field, and the other yielding the null surfaces that are needed to give meaning to the integrals in the first part. The two parts are strongly coupled.

The first set of equations is equivalent to the D'Adhemar version of the self-dual and the anti-self-dual Yang–Mills equations for the self-dual and anti-self-dual parts of an O(3, 1) connection, respectively, on a space–time manifold $\mathcal{M}$ with an arbitrary asymptotically flat metric. The basic idea is to treat a special case of the O(3, 1) Yang–Mills connection so that it agrees with that of the background gravitational connection. This is done by imposing very restrictive conditions on the Yang–Mills data.

The second set of equations determines the light cones of the manifold $\mathcal{M}$. For a *given* metric, one could have obtained this information by integrating the null geodesic equation (see for example [6]). However, in our case where the metric is as yet undetermined, we derive differential equations for the characteristic surfaces that are coupled to the O(3, 1) Yang–Mills field equations.

The above two parts, which look hopelessly intertwined at first, nevertheless give results that are tractable and comprehensible. The exact equations, while complicated, are in

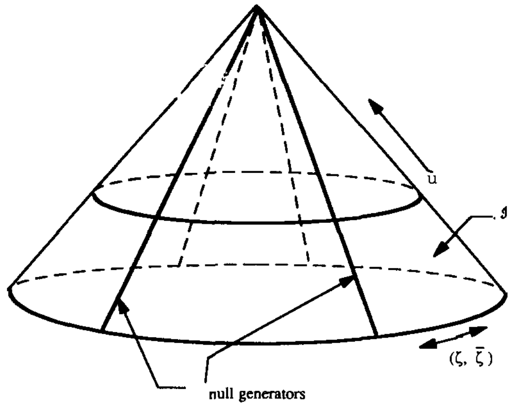

Fig. 1. Bondi coordinates of $\mathcal{I}$.

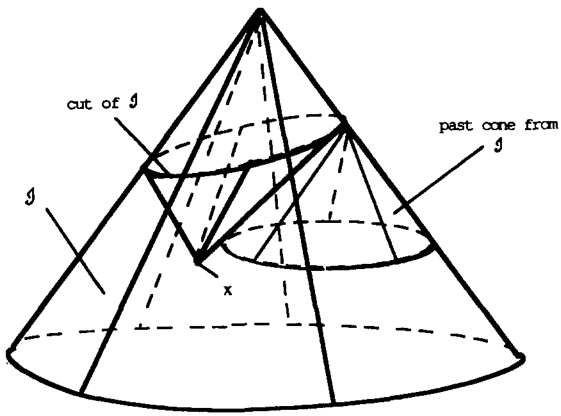

Fig. 2. $\mathcal{I}$, a cut of $\mathcal{I}$ and a past cone of $\mathcal{I}$.

principle solvable since asymptotically flat space–times exist. An attractive feature of this approach is that the equations are amenable to an approximation scheme wherein the nth order correction to the field is expressible as D'Adhemar integrals in terms of lower order.

As the material of this paper is not at all close to mainstream ideas, we believe that it might be appropriate first to give a broad perspective before going into the details.

There are two different sets of ideas that we try to weld together. The first is based on the use of characteristic surfaces on arbitrary asymptotically flat Lorentzian space–times. We make essential use of the future light cones N_x from arbitrary space–time points x^a and their intersections C_x with future null infinity $\mathcal{I}^+$ (referred to as the light cone cuts)—as well as the past light cones from points on $\mathcal{I}^+$. See Figs. 1 and 2. These characteristic surfaces are used to give the D'Adhemar-like reformulation of the Yang–Mills equations. Using Bondi coordinates $(u, \zeta, \bar{\zeta})$ on $\mathcal{I}^+$, C_x is described by a function we call the light cone cut function on $\mathcal{I}^+$of the form $u = Z(x^a, \zeta, \bar{\zeta})$, parametrized by x^a. The cut function plays a dual role: for fixed interior point x^a it describes the "cut" of $\mathcal{I}^+$, and for fixed point

$(u, \zeta, \bar{\zeta})$ on $\mathcal{I}^+$, it describes the past light cone of that point as x^a is varied. From the latter point of view the cut function becomes a three-parameter family of characteristic surfaces. The conformal structure of the space–time can be recovered from knowledge of the cut function, i.e., the conformal metric can be explicitly written in terms of derivatives of Z. The specific metric in the conformal class is chosen so that, in a special null coordinate system, one component of the metric (g^{01}) is taken as one. Geometrically this makes one of the coordinates an affine parameter. More specifically, the conformal metric (picking out a specific member in the class) can be completely expressed in terms of Λ, defined from Z by $\Lambda = \eth^2 Z$. The quantity Λ becomes one of our basic variables. These ideas are developed in Section 2.

The second set of ideas of concern to us (see Section 3) is based on the fact that, in some sense, the O(3, 1) Yang–Mills equations contain, as a special class of solutions, the vacuum Einstein equations. The full Yang–Mills curvature F can be broken into four pieces; self-dual and anti-self-dual on the internal indices as well as self-dual and anti-self-dual on the space–time indices, so that

$$F =^+ F^+ +^+ F^- +^- F^+ +^- F^-, \tag{1}$$

where the $+, -$ symbols in front refer to space–time indices and afterwards to internal duality. The Einstein equations are almost encoded [7] into the algebraic statement that ${}^+F^- =^- F^+ = 0$ needing for completion only an appropriate restriction of the data. The basic idea is to think of solving these equations (for the self-dual and the anti-self-dual fields ${}^+F^+ and\ {}^-F^-$ and their respective connections) on a given asymptotically flat vacuum background metric, then introducing a soldering form and restrict the data so that it coincides with the background connection data. An important technical device is to give these equations a D'Adhemar-like formulation which, as was mentioned earlier, can be done for Yang–Mills fields. The tool for this is the use of the holonomy operator H (around special null paths) as the primitive variable for the Yang–Mills fields. The Bianchi identities, equivalent (because of the self- and anti-self-duality) to the Yang–Mills field equations become one set of our final equations. They are symbolically written as equations for H as

$$\mathcal{B}_\beta(H, A) = 0 \tag{2}$$

with A being the restricted characteristic data.

When the soldering form (a tetrad field) is introduced, we can obtain a set of relationships between the space–time variable $\Lambda \equiv \eth^2 Z$, and H, that we refer to as the $\Lambda(H)$-relations given symbolically by

$$\Lambda = \Lambda(H). \tag{3}$$

Our third set of equations, referred to as the "field equations", is simply the algebraic equations

$${}^+F^- =^- F^+ = 0, \tag{4}$$

expressed as functions of the components of the holonomy operator, symbolically written as

$$\mathcal{D}(H) = 0. \tag{5}$$

Our final task is two-fold:

(1) to eliminate the holonomy variables from these three relationships and be left with equations *only* for the characteristic surfaces, i.e., equations for Z (Section 3.5). These equations constitute the conformal Einstein equations and yield an equation that we refer to as the light cone cut equation (LCCE) which involves only Λ and the free data; and

(2) to find an equation, derivable from Eq. (5), for the conformal factors that converts the conformal metric to a vacuum metric.

In Section 4 we discuss the linearization of the theory and a perturbation scheme.

2. Light cone cuts

Let us consider a real, four-dimensional, asymptotically flat space-time $\mathcal{M}$ with a conformal (unphysical) metric g_{ab} and future null boundary $\mathcal{I}^+$. We use the Bondi coordinates $(u, \zeta, \bar{\zeta})$ to coordinatize $\mathcal{I}^+$. A coordinate grid of the (u = const.) and the ($\zeta, \bar{\zeta}$ = const.) surfaces on $\mathcal{I}^+$is shown in Fig. 1.

Consider the future light cone N_x emanating from an internal point x^a. A relation among the coordinates $(u, \zeta, \bar{\zeta})$ such as $u = f(\zeta,\bar{\zeta})$ describes locally a 2-surface which we refer to as a 'cut'. The intersection of N_x with $\mathcal{I}^+$ is a special cut, the 'light cone cut' (Fig. 2), given by

$$u = Z(x^a, \zeta, \bar{\zeta}). \tag{6}$$

We refer to $Z(x^a, \zeta, \bar{\zeta})$ as the light cone cut function. The function $Z(x^a, \zeta, \bar{\zeta})$ is our basic variable. The conformal metric, i.e., a specific one in the conformal class, can be given as an explicit function of Z [8].

One could also interpret Eq. (6) as a description of the *past* light cone of the point $(u, \zeta, \bar{\zeta})$ on $\mathcal{I}^+$(also shown in Fig. 2). That is, keeping u, ζ and $\bar{\zeta}$ fixed, if we vary x^a, this equation is the locus of all space–time points x^a that are null connected to the point $(u, \zeta, \bar{\zeta})$, which by definition is the past light cone of $(u, \zeta, \bar{\zeta})$ on $\mathcal{I}^+$. From this important observation, we see that for fixed values of $(u, \zeta, \bar{\zeta})$, $\nabla_a Z$ is a null covector, i.e.,

$$g^{ab} Z_{,a}(x, \zeta, \bar{\zeta}) Z_{,b}(x, \zeta, \bar{\zeta}) = 0. \tag{7}$$

Thus it follows that $Z_{,a}(x, \zeta, \bar{\zeta})$ at the point x^a sweeps out the null cone at that point as we vary ζ and $\bar{\zeta}$. Although is has been discussed elsewhere [8] and we will not go into the details, it is from Eq. (7), by taking several ζ and $\bar{\zeta}$ derivatives, the entire conformal metric can be reconstructed completely in terms of Z or more specifically $\Lambda \equiv \eth^2 Z$.

The sphere of null directions at the point x^a is coordinatized by ζ and $\bar{\zeta}$. We use these coordinates for S^2 instead of the usual (θ, ϕ) because covariant differentiation on the sphere, which appears in many of the equations, takes on a particularly simple form in terms of $(\zeta, \bar{\zeta})$. We introduce the operators $\eth$ and $\bar{\eth}$ [9],

$$\eth\alpha_s = (1+\zeta\bar{\zeta})^{1-s}\frac{\partial}{\partial\zeta}[(1+\zeta\bar{\zeta})^s\alpha_s] \tag{8}$$

and

$$\bar{\eth}\alpha_s = (1+\zeta\bar{\zeta})^{1+s}\frac{\partial}{\partial\bar{\zeta}}[(1+\zeta\bar{\zeta})^{-s}\alpha_s], \tag{9}$$

which operate on the quantity α_s defined on the sphere, where s, the spin weight, is assigned according to how α transforms under a specific transformation. (See [10] for more about this transformation and properties of $\eth$ and $\bar{\eth}$.) It helps to think of $\eth$ and $\bar{\eth}$ loosely as differentiation with respect to ζ and $\bar{\zeta}$ respectively.

Assuming that we know the light cone cut function, and with the above definitions of $\eth$ and $\bar{\eth}$, the following set of quantities is well defined:

$$u = Z(x^a, \zeta, \bar{\zeta}), \tag{10}$$
$$\omega = \eth Z(x^a, \zeta, \bar{\zeta}), \tag{11}$$
$$\bar{\omega} = \bar{\eth} Z(x^a, \zeta, \bar{\zeta}), \tag{12}$$
$$R = \eth\bar{\eth} Z(x^a, \zeta, \bar{\zeta}), \tag{13}$$

with $(u, R, \omega, \bar{\omega})$ having, respectively, spin-weights $(0, 0, 1, -1)$. This set defines a *sphere's worth* of coordinate transformations on the space–time parametrized by $(\zeta, \bar{\zeta})$. Let

$$(\theta^0, \theta^1, \theta^+, \theta^-) \equiv (u, R, \omega, \bar{\omega}). \tag{14}$$

With this notation, the $(\zeta, \bar{\zeta})$-dependent coordinate transformation can be written as

$$\theta^i = \theta^i(x^a, \zeta, \bar{\zeta}), \tag{15}$$

where the indices i, j will take on the values $\{0, 1, +, -\}$.

We now construct

$$\eth^2 Z \equiv \Lambda(x^a, \zeta, \bar{\zeta}). \tag{16}$$

The quantity Λ and its conjugate can be expressed as functions of the θ^i by inverting the transformation (15) and eliminating the x^a to obtain

$$\Lambda(\theta^i, \zeta, \bar{\zeta}) \equiv \eth^2 Z. \tag{17}$$

Later we show that the (conformal) Einstein equations can be encoded into the choice of $\Lambda(\theta^i, \zeta, \bar{\zeta})$. Note that if $\Lambda(\theta^i, \zeta, \bar{\zeta})$ is given with θ^i replaced by Eqs. (10)–(13), Eq. (17) is a second-order differential equation for Z; the x^a are the constants of integration. Also if $\eth$ is applied to a function $F(\theta^i, \zeta, \bar{\zeta})$, it means the total derivative, treating the θ^i as functions of $(\zeta, \bar{\zeta})$.

With the θ^i coordinate system, we have a one-form basis:

$$\mathrm{d}\theta^i = \theta^i{}_a\,\mathrm{d}x^a, \quad \text{with}\, \theta^i{}_{,a} = \theta^i{}_a \tag{18}$$

and the dual basis

$$\frac{\partial}{\partial\theta^i} = \theta^a{}_i \frac{\partial}{\partial x^a}, \tag{19}$$

with the relations

$$\theta^i{}_a \theta^a{}_j = \delta^i{}_j, \tag{20}$$

$$\theta^i{}_a \theta^b{}_i = \delta_a{}^b. \tag{21}$$

Any vector V_a can be written as $V_a = V_i \theta^i{}_a$ and in particular we have

$$\Lambda_{,a} = \Lambda_{,i}\theta^i{}_a = \Lambda_{,0} Z_a + \cdots \tag{22}$$

The $\Lambda_{,i}$ play a basic role in what follows.

The metric, expressed in the θ^i coordinates, i.e., $g^{ij} = \theta^i{}_a \theta^j{}_b g^{ab}$ takes the form [7],

$$(g^{ij}) = \Omega^2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & h^{11} & h^{1+} & h^{1-} \\ 0 & h^{1+} & h^{++} & -1 \\ 0 & h^{1-} & -1 & h^{--} \end{pmatrix}, \tag{23}$$

with the h's explicit functions of Λ.

Note also that $\eth\theta^i{}_a$ can also be expressed as a linear combination of the $\theta^i{}_a$, i.e., as

$$\eth\theta^i{}_a = T^i{}_j \theta^j{}_a. \tag{24}$$

For example, from $\theta^i{}_a \equiv \{Z_a, \eth\bar{\eth} Z_a, \eth Z_a, \bar{\eth} Z_a\}$, we obtain that

$$\eth\theta^0{}_a = \theta^+_a, \tag{25}$$

$$\eth\theta^+{}_a = \Lambda_{,a} = \Lambda_{,i}\theta^i{}_a, \tag{26}$$

$$\eth\theta^-{}_a = \theta^1{}_a, \tag{27}$$

$$\eth\theta^1{}_a = T^1{}_i \theta^i{}_a, \tag{28}$$

with

$$T^1{}_i = (1/q)\Big[\Lambda_1(\eth\bar{\Lambda}_i + \bar{\Lambda}_0 \delta^+{}_i + \bar{\Lambda}_- \delta^1{}_i + \bar{\Lambda}_+ \Lambda_i - 2\delta^-{}_i) + \bar{\eth}\Lambda_i + \Lambda_0 \delta^-{}_i + \Lambda_+ \delta^1{}_i + \Lambda_- \bar{\Lambda}_i - 2\delta^+{}_i\Big],$$

and

$$q = (1 - \Lambda_1 \bar{\Lambda}_1). \tag{29}$$

Likewise,

$$\bar{\eth}\theta^i{}_a = \overline{T}^i_j \theta^j{}_a \tag{30}$$

with $\overline{T}^i_j$ obtained from the complex conjugate coefficients of $T^i{}_j$. The explicit form of these matrices with row and column indices 0, 1, +, − are:

$$T^i{}_j = \begin{pmatrix} 0 & 0 & 1 & 0 \\ T^1{}_0 & T^1{}_1 & T^1{}_+ & T^1{}_- \\ \Lambda_0 & \Lambda_R & \Lambda_+ & \Lambda_- \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{31}$$

$$\overline{T}^i_j = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \overline{T}^1{}_0 & \overline{T}^1{}_1 & \overline{T}^1{}_+ & \overline{T}^1{}_- \\ 0 & 1 & 0 & 0 \\ \bar{\Lambda}_0 & \bar{\Lambda}_1 & \bar{\Lambda}_+ & \bar{\Lambda}_- \end{pmatrix}. \tag{32}$$

3. Holonomy and Einstein equations

3.1. O(3, 1) *Yang–Mills equations*

We begin with a real four-dimensional Lorentzian manifold $\mathcal{M}$ where we *assume* that the light cone structure of the previous section is known and study an O(3, 1) Yang–Mills field. The idea is to (roughly) think of the associated vector bundle as being the tangent bundle of the space–time, though of course a soldering form is needed (see Section 3.2) to make this precise.

The connection one-form, which is antisymmetric in the Lorentzian indices, $\hat{i}, \hat{j}, \ldots,$ can be decomposed into its self-dual and anti-self-dual parts,

$$\gamma_a{}^{\hat{i}}{}_{\hat{j}} = \gamma_a^{+\hat{i}}{}_{\hat{j}} + \gamma_a^{-\hat{i}}{}_{\hat{j}}, \tag{33}$$

where self-dual and anti-self-dual are defined by

$$\gamma_a^{\pm\hat{i}}{}_{\hat{j}} = \tfrac{1}{2}\left(\gamma_a{}^{\hat{i}}{}_{\hat{j}} \mp i\gamma_a^{*\hat{i}}{}_{\hat{j}}\right), \tag{34}$$

and duality by

$$\gamma_a^{*\hat{i}\hat{j}} = \tfrac{1}{2}\epsilon^{\hat{i}\hat{j}}{}_{\hat{k}\hat{\ell}}\gamma_a{}^{\hat{k}\hat{\ell}}, \tag{35}$$

where $\epsilon^{\hat{i}\hat{j}}{}_{\hat{k}\hat{l}}$ is the alternating symbol with $\epsilon_{0123} = -1$. The curvature tensor can be similarly decomposed as,

$$F_{ab}{}^{\hat{i}}{}_{\hat{j}} = F_{ab}^{+\hat{i}}{}_{\hat{j}} + F_{ab}^{-\hat{i}}{}_{\hat{j}}, \tag{36}$$

where the self-dual and anti-self-dual parts of F are constructed from the self-dual and anti-self-dual connections respectively, i.e.,

$$F_{ab}^{\pm\hat{i}}{}_{\hat{j}} = \nabla_{[a}\gamma^{\pm\hat{i}}_{b]\hat{j}} + [\gamma_a^{\pm}, \gamma_b^{\pm}]^{\hat{i}}{}_{\hat{j}}. \tag{37}$$

With the above decompositions, the Bianchi identities and the field equations become

$$\nabla_{[c} F^{\pm}_{ab]}{}^{\hat{i}}{}_{\hat{j}} + [\gamma^{\pm}_{[c}, F^{\pm}_{ab]}]^{\hat{i}}{}_{\hat{j}} = 0 \tag{38}$$

and

$$\nabla^{a} F^{\pm\hat{i}}_{ab}{}_{\hat{j}} + [\gamma^{\pm a}, F^{\pm}_{ab}]^{\hat{i}}{}_{\hat{j}} = 0, \tag{39}$$

respectively. One is thus dealing with two independent complex Yang–Mills connections and fields.

It is possible to further decompose each of the two curvature tensors, now on the space–time indices, into its space–time self-dual and anti-self-dual parts, where we have used the existence of the Lorentzian metric. We will refer to space–time duals as left duals and internal duals as right duals. The full curvature then has four terms:

(1) the left and right self-dual part ${}^{+}F^{+}_{ab}$;
(2) the left anti-self-dual and right self-dual part ${}^{-}F^{+}_{ab}$;
(3) the left self-dual and right anti-self-dual part ${}^{+}F^{-}_{ab}$;
(4) the left anti-self-dual and right anti-self-dual part ${}^{-}F^{-}_{ab}$.

Parts (1) and (2) are coupled as are parts (3) and (4), in the sense that they depend respectively on the γ^{+} and γ^{-}.

We next assume that

$${}^{-}F^{+}_{ab} = {}^{+}F^{-}_{ab} = 0. \tag{40}$$

From this the field equations (39) are automatically satisfied via the Bianchi identities (38). Thus we have two independent Yang–Mills fields: a (left) self-dual field ${}^{+}F^{+}_{ab}$ and a (left) anti-self-dual field ${}^{-}F^{-}_{ab}$ satisfying the Bianchi identities (38) and the above condition (40). These two equations are rewritten in terms of a new variable, viz., the holonomy operator, in the next section.

Eq. (38) and (40) *with a special class of data* (on $\mathcal{I}^{+}$) are equivalent to the Einstein equations. See Eq. (49).

3.2. *Holonomy and the Bianchi identities*

The variable that we have been (loosely) referring to as the holonomy operator (associated with the Yang–Mills connection γ_a) and denoting it by H, is more accurately the difference between the holonomy operator associated with an infinitesimal path and the identity. It should really be called the infinitesimal holonomy operator or the differential holonomy operator, though we will continue with the original name. There are two distinctly different sets of paths $\Delta_x(\zeta,\bar{\zeta})$ and $\bar{\Delta}_x(\zeta,\bar{\zeta})$ (with their own holonomies H and $\overline{H}$) defined in the following manner: Consider an interior point x^a of $\mathcal{M}$. For $\Delta_x(\zeta,\bar{\zeta})$ we choose two null rays on the cone N_x that are infintesimally separated, namely, $\ell_x(\zeta,\bar{\zeta})$ and $\ell_x(\zeta + \mathrm{d}\zeta,\bar{\zeta})$, extending from x^a to $\mathcal{I}^{+}$(see Fig. 3). We then form a closed loop by connecting the end points of these two rays on $\mathcal{I}^{+}$. At $\mathcal{I}^{+}$the two-form constructed from this connecting vector and the tangent vector to the geodesics is self-dual. (In Minkowski space the entire path lies

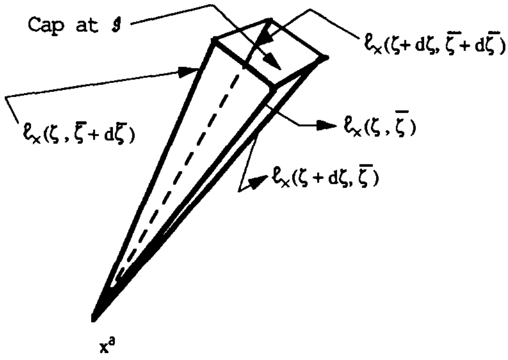

Fig. 3. Volume V.

in a self-dual blade; it is the loss of this property in curved space–time that is the source of non-Huygens propagation for linear rest-mass zero fields.) In a similar manner, one can choose the anti-self-dual triangle $\bar{\Delta}_x(\zeta,\bar\zeta)$, which has $\ell_x(\zeta,\bar\zeta)$ and $\ell_x(\zeta,\bar\zeta+\mathrm{d}\bar\zeta)$ for its sides.

For our O(3, 1) connection, the vectors that are being propagated around the closed loops Δ_x and $\bar\Delta_x$ are thought of (using a soldering form $\lambda^a{}_i(x)$ introduced below) as being in the tangent (or cotangent) bundle. The effect of applying the operators H or $\overline{H}$ to an arbitrary vector V^μ at x^a is, respectively,

$$V'^\mu = V^\nu\left(\delta^\mu{}_\nu + \frac{H^\mu{}_\nu(x^a,\zeta,\bar\zeta)\,\mathrm{d}\zeta}{2P}\right) \tag{41}$$

and

$$V'^\mu = V^\nu\left(\delta^\mu{}_\nu + \frac{\overline{H}^\mu{}_\nu(x^a,\zeta,\bar\zeta)\,\mathrm{d}\bar\zeta}{2P}\right), \tag{42}$$

where the $P=(1+\zeta\bar\zeta)$ is for notational purposes. H takes us from a point on $\mathcal{I}^+$along $\ell_x(\zeta,\bar\zeta)$ down to x^a then back to $\mathcal{I}^+$along $\ell_x(\zeta+\mathrm{d}\zeta,\bar\zeta)$ and finally back to the starting point along a connecting vector on $\mathcal{I}^+$(see Fig. 3). For parallel transport around this loop one obtains the following:

$$H^\mu{}_\nu(x^a,\zeta,\bar\zeta) = A^\mu{}_\nu + (G^{-}1)_\nu{}^b\eth G^\mu{}_b, \tag{43}$$

where $G^\mu{}_a(x^a,\zeta,\bar\zeta)$ is the parallel propagator that takes vectors from the point x^a to $\mathcal{I}^+$along the null geodesic $\ell_x(\zeta,\bar\zeta)$, and $A^\mu{}_\nu$ are the asymptotic values (characteristic data) of the connection γ_a in the direction of the connecting vector on $\mathcal{I}^+$. See [7,11] for a derivation of this result.

It will be convenient (see Eq. (60)) to express the above relation in terms of a given *null* tetrad $\lambda^a{}_i(x)$ (soldering form) defined on $\mathcal{M}$ and $\mathcal{I}^+$with normalization

$$g_{ab}\lambda^a{}_{\hat i}\lambda^b{}_{\hat j} = \eta_{\hat i\hat j} \tag{44}$$

with

$$\eta_{\hat{i}\hat{j}} = \eta^{\hat{i}\hat{j}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{45}$$

and $H^{\hat{i}}{}_{\hat{j}} = H^{\mu}{}_{\nu}\lambda^{\hat{i}}{}_{\mu}\lambda^{\nu}{}_{\hat{j}}$, etc. Tetrad indices are raised and lowered with $\eta^{\hat{i}\hat{j}}$ and $\eta_{\hat{i}\hat{j}}$, respectively. For the tetrad fields $\lambda^{\mu}{}_{\hat{i}}$ at $\mathcal{I}^{+}$we will use the Bondi tetrad $\{\ell_{\mu}, n_{\mu}, m_{\mu}, \overline{m}_{\mu}\}$ associated with the Bondi coordinates.

Returning to the Bianchi identities (38), they can be written in terms of the holonomy operator and the characteristic data A as

$$\bar{\eth}(H-A)^{\hat{i}}{}_{\hat{j}} - \eth(\overline{H}-\bar{A})^{\hat{i}}{}_{\hat{j}} + [H-A, \overline{H}-\bar{A}]^{\hat{i}}{}_{\hat{j}} - [H, \bar{A}]^{\hat{i}}{}_{\hat{j}} - 0. \tag{46}$$

At this point all the functions can be thought of as depending on x^{a} and $\zeta, \bar{\zeta}$; using Eq. (15) they are to be reinterpreted as functions of θ^{i} and $\zeta, \bar{\zeta}$.

A derivation of this result via an integration of (38) over the region V of Fig. 3, can be found in [12]. Using $\eta^{\hat{i}\hat{j}}$ to raise one of the indices, we obtain $H^{\hat{i}\hat{j}}$, $A^{\hat{i}\hat{j}}$, and their complex conjugates. Since these are skew in the $\hat{i}\hat{j}$ indices, they can be separated into self-dual and anti-self-dual parts, yielding

$$H^{\hat{i}\hat{j}} = H^{(-)\hat{i}\hat{j}} + h^{(+)\hat{i}\hat{j}} \quad \text{and} \quad A^{\hat{i}\hat{j}} = A^{(+)\hat{i}\hat{j}} + A^{(-)\hat{i}\hat{j}} \tag{47}$$

and

$$\overline{H}^{\hat{i}\hat{j}} = H^{(+)\hat{i}\hat{j}} + h^{(-)\hat{i}\hat{j}} \quad \text{and} \quad A^{\hat{i}\hat{j}} = A^{(-)\hat{i}\hat{j}} + A^{(+)\hat{i}\hat{j}}. \tag{48}$$

We have three complex non-trivial components of H which we denote by H_{α} and likewise three h_{α}, where $\alpha = \{1,2,3\}$, as shown in Table 1. As for the characteristic data A we choose it to have the very special form:

$$A^{i}_{j} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\dot{\bar{\sigma}}_{\mathrm{B}} \\ -\dot{\bar{\sigma}}_{\mathrm{B}} & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \tag{49}$$

and

$$\bar{A}^{i}_{j} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -\dot{\bar{\sigma}}_{\mathrm{B}} & 1 \\ 1 & -1 & 0 & 0 \\ -\dot{\bar{\sigma}}_{\mathrm{B}} & 0 & 0 & 0 \end{pmatrix}. \tag{50}$$

This form for the A's comes from the demand that the data for the O(3, 1) Yang–Mills field should coincide with that of the background geometry; the asymptotic γ's should be the same as the asymptotic spin coefficients. The σ_{B} is to be the Bondi shear. The self-dual and anti-self-dual parts of H can be written out explicitly as matrices,

Table 1
Notation

New notation	$\hat{i}\hat{j}$ component	Spin weight	New notation	$\hat{i}\hat{j}$ component	Spin weight
H_1	$-H^{(-)0+}$	2	$\overline{H}_1$	$-H^{(+)0-}$	-2
H_2	$-H^{(-)01}$	1	$\overline{H}_2$	$-H^{(+)01}$	-1
H_3	$-H^{(-)1-}$	0	$\overline{H}_3$	$-H^{(+)1+}$	0
h_1	$-h^{(-)0+}$	0	$\bar{h}_1$	$-h^{(+)0-}$	0
h_2	$-h^{(-)01}$	-1	$\bar{h}_2$	$-h^{(+)01}$	1
h_3	$-h^{(-)1-}$	-2	$\bar{h}_3$	$-h^{(+)1+}$	2

$$H^{(-)\hat{i}\hat{j}} = \begin{pmatrix} 0 & \frac{1}{2}(H^{01} - H^{+-}) & H^{0+} & 0 \\ -\frac{1}{2}(H^{01} - H^{+-}) & 0 & 0 & H^{1-} \\ -H^{0+} & 0 & 0 & -\frac{1}{2}(H^{01} - H^{+-}) \\ 0 & -H^{1-} & -\frac{1}{2}(H^{01} - H^{+-}) & 0 \end{pmatrix} \tag{51}$$

and

$$h^{(+)\hat{i}\hat{j}} = \begin{pmatrix} 0 & \frac{1}{2}(H^{01} + H^{+-}) & 0 & H^{0+} \\ -\frac{1}{2}(H^{01} + H^{+-}) & 0 & H^{1+} & 0 \\ 0 & -H^{1+} & 0 & \frac{1}{2}(H^{01} + H^{+-}) \\ -H^{0-} & 0 & -\frac{1}{2}(H^{01} + H^{+-}) & 0 \end{pmatrix}, \tag{52}$$

and likewise for $\overline{H}$.

In order to raise and lower the $\hat{i}$, $\hat{j}$ indices, we have to use the null-coordinate version of the Minkowski metric $\eta_{\hat{i}\hat{j}}$ and $\eta^{\hat{i}\hat{j}}$. For example,

$$H^0{}_1 = H^{00} = 0, \quad \text{by skew symmetry,} \tag{53}$$

$$H^0{}_+ = -H^{0-}. \tag{54}$$

Also note that the complex conjugate of a quantity, say, $\overline{H^{0-}}$ gives us $\overline{H}^{0+}$: the 0 and 1 indices are insensitive to complex conjugation, while + and − are interchanged. Table 1 displays our notation.

With this notation the holonomy Bianchi identities (46) become:

$$\eth h_1 - \bar{\eth} H_1 + 2h_1 H_1 - 2H_1 h_2 + 2H_2 = 0, \tag{55}$$

$$\eth h_2 - \bar{\eth} H_2 + h_3 H_1 - H_3(h_1 + 1) + h_1 = -\dot{\bar{\sigma}}_{\mathrm{B}} H_1, \tag{56}$$

$$\eth h_3 - \bar{\eth} H_3 + 2h_2(H_3 - 1) - 2H_2 h_3 = -\eth\dot{\bar{\sigma}}_{\mathrm{B}} + 2\dot{\bar{\sigma}}_{\mathrm{B}} H_2, \tag{57}$$

where we have the data or "driving" terms (i.e., those that involve $\bar{\sigma}_B$) on the right. Eq. (46) actually contains this triplet as well as its conjugate—they are the result of taking the self-dual and anti-self-dual parts of Eq. (46). This is the first set of equations that was given symbolically as Eq. (2) in Section 1. (Note that the first equation is algebraic in h_2 and H_2, and the second is algebraic in h_3 and H_3. We will exploit this structure in Section 3.4 where we study this set of equations in more detail.)

For clarity we remark that Eqs. (55)–(57) follow directly from the O(3, 1) Yang–Mills field equations and Bianchi identities with the special data, Eq. (49). No use is made yet of the soldering form.

3.3. The $\Lambda(H)$-relations

In the previous section we defined the O(3, 1) propagator $G^{\mu}{}_{a}(x^a, \zeta, \bar{\zeta})$ (converted via the soldering form to act on space–time vectors) which takes vectors from a point on $\mathcal{I}^+$ to an interior point x^a along a null geodesic $\ell_x(\zeta, \bar{\zeta})$. In particular, the tetrad fields $\{\lambda^{\hat{i}}{}_{\mu}, \lambda^{\mu}{}_{\hat{i}}\}$ that are defined on $\mathcal{I}^+$ can be parallel propagated in this manner to the point x^a. The parallel propagated tetrad fields at x^a are given by

$$e^{\hat{i}}{}_{a} = \lambda^{\hat{i}}{}_{\mu} G^{\mu}{}_{a}, \tag{58}$$

where $\lambda^{\hat{i}}{}_{\mu}$ represents the tetrad $\{\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu\}$ on $\mathcal{I}^+$. We use the following notation for the parallel propagated tetrad at x^a:

$$e^{\hat{i}}{}_{a} \equiv \{e^{0}{}_{a}, e^{1}{}_{a}, e^{+}{}_{a}, e^{-}{}_{a}\} \equiv \{\ell_a, n_a, m_a, \overline{m}_a\}. \tag{59}$$

If we take two tetrad vectors on $\mathcal{I}^+$, one at $(\zeta, \bar{\zeta})$ and the other (on the same cut, associated with x^a) at $(\zeta + \mathrm{d}\zeta, \bar{\zeta})$, and parallel propagate both to the point x^a, the difference in terms of the holonomy operator H and the free data A [7] is given by

$$e^{a}{}_{\hat{j}} \eth e^{\hat{i}}{}_{a} = H^{\hat{i}}{}_{\hat{j}} - A^{\hat{i}}{}_{\hat{j}}, \tag{60}$$

$$e^{a}{}_{\hat{j}} \bar{\eth} e^{\hat{i}}{}_{a} = \overline{H}^{\hat{i}}{}_{\hat{j}} - \bar{A}^{\hat{i}}{}_{\hat{j}}. \tag{61}$$

We have introduced three different tetrad bases: $\lambda^{\hat{i}}{}_{a}$, $\lambda^{\hat{i}}{}_{\mu}$, and $e^{\hat{i}}{}_{a}$. The first of these $\lambda^{\hat{i}}{}_{a}$ is an arbitrary tetrad given at x^a and therefore independent of ζ and $\bar{\zeta}$. The second $\lambda^{\hat{i}}{}_{\mu}$ is a $(\zeta, \bar{\zeta})$-dependent Bondi tetrad given on $\mathcal{I}^+$. And the third $e^{\hat{i}}{}_{a}$ is the result of parallel propagating the $\lambda^{\hat{i}}{}_{\mu}$ in from $\mathcal{I}^+$ to x^a along $\ell_x(\zeta, \bar{\zeta})$, and therefore is also $(\zeta, \bar{\zeta})$-dependent.

From Section 3, we have at our disposal the θ^i coordinates and the associated bases $\theta^{i}{}_{a}$ and θ^{ai} at x^a. Since any vector at x^a can be expressed as a linear combination of either of the two sets $\{\theta^{i}{}_{a}\}$ and $\{e^{\hat{i}}{}_{a}\}$, we can go from one to the other, using the invertible transformation

$$\theta^{i}{}_{a} = \Sigma^{i}{}_{\hat{i}} e^{\hat{i}}{}_{a}. \tag{62}$$

We choose one of the 'legs' of the parallely propagated tetrad vectors $e^{0}{}_{a}$ equal to the vector $\nabla_a Z$, i.e.,

$$e^{0}{}_{a} \equiv \ell_a = Z_{,a} \equiv \theta^{0}{}_{a}. \tag{63}$$

Using this assumption and Eqs. (60) and (61) with our notation from Table 1, the transformation matrix $\Sigma^i{}_{\hat{i}} \equiv \boldsymbol{\Sigma}$ and its inverse $\Sigma^{\hat{i}}{}_i \equiv \boldsymbol{\Sigma}^{-1}$ can be calculated (by application of $\eth$ and $\bar{\eth}$ to (60) and (61)) and written explicitly as functions of the H's and h's [12]. $\boldsymbol{\Sigma}$ written as matrix is given by

$$\left(\Sigma^i{}_{\hat{i}}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \Sigma^1{}_0 & \Sigma^1{}_1 & \Sigma^1{}_+ & \Sigma^1{}_- \\ -(H_2 + \bar{h}_2) & 0 & (1 + \bar{h}_1) & H_1 \\ -(\overline{H}_2 + h_2) & 0 & \overline{H}_1 & (1 + h_1) \end{pmatrix} \tag{64}$$

with

$$\begin{aligned} \Sigma^1{}_0 &= -\eth(\overline{H}_2 + h_2) + (\overline{H}_2 + h_2)(H_2 + \bar{h}_2) \\ &\quad + \overline{H}_1(\dot{\sigma}_\mathrm{B} + \bar{h}_3) + (1 + h_1)(H_3 - 1); \\ \Sigma^1{}_1 &= (1 + h_1)(1 + \bar{h}_1) + H_1\overline{H}_1; \\ \Sigma^1{}_+ &= \eth\overline{H}_1 - (\overline{H}_2 + h_2)(1 + \bar{h}_1) - \overline{H}_1(H_2 - \bar{h}_2); \\ \Sigma^1{}_- &= \eth\bar{h}_1 + (H_2 - \bar{h}_2)(1 + h_1) - H_1(\overline{H}_2 + h_2). \end{aligned}$$

Using $\boldsymbol{\Sigma}$ and its conjugate, and the Eqs. (24), i.e.,

$$\eth\theta^i{}_a = T^i{}_j\theta^j{}_a \tag{65}$$

and their conjugates, the T's and $\overline{T}$'s of Section 2 can be expressed directly in terms of the H's and h's via the $\Sigma^i{}_{\hat{i}}$ by

$$\boldsymbol{T} = (\eth\boldsymbol{\Sigma}) \cdot \boldsymbol{\Sigma}^{-1} + \boldsymbol{\Sigma} \cdot (\boldsymbol{H} - \boldsymbol{A}) \cdot \boldsymbol{\Sigma}^{-1}, \tag{66}$$

where the boldface represents the corresponding matrices [12]. Since $T^i{}_j$ was shown in Eq. (31) to have the form

$$\boldsymbol{T} \equiv (T^i{}_j) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ T^1{}_0 & T^1{}_1 & T^1{}_+ & T^1{}_- \\ \Lambda_0 & \Lambda_R & \Lambda_+ & \Lambda_- \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{67}$$

we obtain from Eqs. (65) and (66) the $\Lambda(H)$-relations:

$$\Lambda_1 \equiv \Lambda_R = \frac{2(\bar{h}_1 + 1)H_1}{(\bar{h}_1 + 1)(h_1 + 1) + \overline{H}_1 H_1}, \tag{68}$$

$$\Lambda_+ = W - \tfrac{1}{2}\left(\Lambda_1\overline{W} + \bar{\eth}\Lambda_1 - \eth\ln q\right), \tag{69}$$

$$\Lambda_- = \tfrac{1}{2}\left[\eth\Lambda_1 - \Lambda_1\frac{\eth(H_1\overline{H}_1)}{H_1\overline{H}_1} - \Lambda_1\eth\ln\left(\frac{q}{1 - \sqrt{q}}\right)\right], \tag{70}$$

$$\Lambda_0 = \dot{\sigma}_\mathrm{B} + \tfrac{1}{2}\Lambda_1 - \tfrac{1}{4}\eth\bar{\eth}\Lambda_1 + O(H^2), \tag{71}$$

where W is defined by

$$W \equiv \frac{(\eth \Lambda_1 - 2\Lambda_-)}{\Lambda_1} - \eth \ln q, \tag{72}$$

with

$$q \equiv (1 - \Lambda_1 \bar{\Lambda}_1). \tag{73}$$

See [12,3] for a more complete derivation of these equations. The precise form of Eq. (71) turns out to be rather lengthy and is given in Appendix A of [12].

It is perhaps worth remarking that, from Eq. (62), the full space–time metric can be expressed in terms of the H's and h's. Since the conformal structure is encoded into Λ, Eqs. (68)–(71) express the extraction of the Λ's from the full metric.

3.4. The holonomy field equations

In this section, we describe an important result obtained earlier [7], wherein the vacuum Einstein equations in the form

$$^{+}F^{-}_{ab} = 0 \tag{74}$$

were imposed on the holonomy operator. The goal is to obtain field equations for the holonomy operator, and therefore we first need a relationship between the curvature tensor and the holonomy operator.

It can be shown [7], using a non-Abelian version of Stokes' theorem, that the holonomy operator satisfies

$$H = \int_{s_0}^{\infty} (F^{+}_{ab} + F^{-}_{ab})\ell^a M^b \, \mathrm{d}s = h^{(+)} + H^{(-)} \tag{75}$$

and

$$\overline{H} = \int_{s_0}^{\infty} (F^{+}_{ab} + F^{-}_{ab})\ell^a \overline{M}^b \, \mathrm{d}s = H^{(+)} + h^{(-)}, \tag{76}$$

where the H and h with the plus and minus signs were defined in Section 3.2, s is an affine parameter along the generators of the light cone at x^a with $s = s_0$ at x^a. M^a and $\overline{M}^a$ are connecting vectors between the long legs of Δ_x and $\bar{\Delta}_x$, respectively. See [7] for more details.

The two Eqs. (75) and (76) can be inverted [7] to obtain F^{-}_{ab} and F^{+}_{ab} explicitly in terms of the components of the holonomy operator. Specifically F^{-}_{ab} and F^{+}_{ab} can be expressed as two derivatives of the H's with respect to $\theta^0 = R$. Now by using this expression for the curvature in $^{+}F^{-}_{ab} = 0$ one obtains a differential relationship between the different components of H and $\overline{H}$ given by (see [7])

$$[q^{-1} h_{\alpha,R}]_{,R} + \delta[q^{-1} \Lambda_{,R} h_{\alpha,R}]_{,R} = [q^{-1} \overline{\Lambda}_{,R} H_{\alpha,R}]_{,R} + \delta[q^{-1} H_{\alpha,R}]_{,R}, \tag{77}$$

where

$$\delta = \frac{\sqrt{q}-1}{\Lambda_{,R}} \quad \text{and} \quad q = 1 - \Lambda_R \bar{\Lambda}_R, \tag{78}$$

with $R = \eth\bar{\eth}Z = \theta^0$.

Note that the Eqs. (77), *which we refer to as the field equations*, are three linear relations between the H and h with coefficients depending on Z via Λ_1. They are our final set of equations. In the next section, we will use these with the Bianchi identities and the $\Lambda(H)$-relations to simplify the overall structure of the theory.

Again, we remark, for clarity, that the dependent variables are now functions of θ^i and $\zeta, \bar{\zeta}$.

3.5. The full theory

We have at this point three sets of equations which are equivalent to the asymptotically flat vacuum Einstein equations:

– the three holonomy Bianchi identities:

$$\eth h_1 - \bar{\eth} H_1 + 2h_1 H_1 - 2H_1 h_2 + 2H_2 = 0, \tag{79}$$

$$\eth h_2 - \bar{\eth} H_2 + h_3 H_1 - H_3(h_1 + 1) + h_1 = \dot{\sigma}_B H_1, \tag{80}$$

$$\eth h_3 - \bar{\eth} H_3 + 2h_2(H_3 - 1) - 2H_2 h_3 = -\eth\dot{\sigma}_B + 2\dot{\sigma}_B H_2, \tag{81}$$

– the holonomy field equations:

$$[q^{-1}h_{\alpha,R}]_{,R} + \delta[q^{-1}\Lambda_{,R}h_{\alpha,R}]_{,R} = [q^{-1}\bar{\Lambda}_{,R}H_{\alpha,R}]_{,R} + \delta[q^{-1}H_{\alpha,R}]_{,R}, \tag{82}$$

– the $\Lambda(H)$-relations:

$$\Lambda_1 \equiv \Lambda_R = \frac{2(\bar{h}_1+1)H_1}{(\bar{h}_1+1)(h_1+1) + \overline{H}_1 H_1}, \tag{83}$$

$$\Lambda_+ = W - \tfrac{1}{2}(\Lambda_1 \overline{W} + \bar{\eth}\Lambda_1 - \eth \ln q), \tag{84}$$

$$\Lambda_- = \tfrac{1}{2}\left[\eth\Lambda_1 - \Lambda_1 \frac{\eth(H_1\overline{H}_1)}{H_1\overline{H}_1} - \Lambda_1 \eth \ln\left(\frac{q}{1-\sqrt{q}}\right)\right], \tag{85}$$

$$\Lambda_0 = \dot{\sigma}_B + \tfrac{1}{2}\Lambda_1 - \tfrac{1}{4}\eth\bar{\eth}\Lambda_1 + O(H^2), \tag{86}$$

and their conjugates. The first set involves the H's and h's and the data only, while the other two involve the H's and h's and the Λ_i. We think of the H's and h's as describing the "field" and the Λ_i, the "background". In this section we study the above equations and show how they can be reduced to a smaller and simpler set.

This simplification procedure starts with the following important observation: Eq. (83) and its complex conjugate are two *algebraic* relations between the six quantities, viz., Λ_R and $\bar{\Lambda}_R, H_1, \overline{H}_1, h_1$ and $\bar{h}_1$. Solving for h_1 and $\bar{h}_1$ algebraically yields,

$$h_1 = -(H_1/\bar{\delta}) - 1, \tag{87}$$

$$\bar{h}_1 = -(\overline{H}_1/\delta) - 1, \tag{88}$$

where δ is given by

$$\delta = \frac{\sqrt{q}-1}{\Lambda_{,R}} \quad \text{and} \quad q = 1 - \Lambda_R \bar{\Lambda}_R. \tag{89}$$

Thus h_1 and $\bar{h}_1$ are not independent quantities and Eqs. (87) and (88) allow us to eliminate them completely. This *algebraic* structure extends to the other H's and h's. We are able to solve, from Eqs. (79) and (80), for all the h's and completely eliminate them from the remainder of the analysis.

The algebraic relationships between H_α and h_α for all $\alpha = \{1, 2, 3\}$ takes the form

$$h_\alpha = -(H_\alpha/\bar{\delta}) + G_\alpha, \tag{90}$$

where G_α is a function of Λ_R and the preceding H's. See [12] for explicit expressions for the G_α. The important consequence of the above structure is that after eliminating the h's from the field equations (82) we obtain a very attractive set of three differential equations for H_α:

$$\frac{\partial^2 H_\alpha}{\partial R^2} + X\frac{\partial H_\alpha}{\partial R} + YH_\alpha = \mu\frac{\partial^2 G_\alpha}{\partial R^2} + \nu\frac{\partial G_\alpha}{\partial R}, \tag{91}$$

where X, Y, μ and ν are functions only of Λ_R and $\bar{\Lambda}_R$.

These three equations are linear with the only difference between them being in the homogeneous terms: the first of them, the equation for H_1, is homogeneous with the coefficients depending only on Λ_1 (a function of R), while for the next pair the inhomogeneous terms are driven by the solutions of the previous ones. From this structure, if the homogeneous equation could be solved, the remaining ones could be solved by quadratures.

A possible point of confusion concerning Eq. (91) could be where are other θ^i derivatives (other than $R = \theta^1$) in such a basic equation (which is essentially Eq. (40)). The answer lies in the fact that as the $(\zeta, \bar{\zeta})$ vary, the R derivative spans a sphere's worth of (null) directions.

To summarize, so far we have used the information contained in two (of the three) Bianchi identities, one of the (Λ, H)-equation, and all the field equations. The third Bianchi identity has not yet played any role. *Assuming* that we could solve Eq. (91) for H_1 as a function (functional) of Λ_1 we would have all the $H_\alpha = H_\alpha[\Lambda_1, \bar{\Lambda}_1]$ and their complex conjugates. These expressions, which would be explicit functions of Λ_1 and $\bar{\Lambda}_1$, could then be substituted in the third Bianchi identity. The resulting equation would involve only Λ, $\bar{\Lambda}$, and the data. The single resulting equation, involving only Λ_1 and $\bar{\Lambda}_1$, i.e.,

$$\begin{aligned} &\eth(H_3/\bar{\delta} + G_3) - \bar{\eth}H_3 + 2(H_2/\bar{\delta} + G_2)(H_3 - 1) - 2H_2(H_3/\bar{\delta} + G_3) \\ &\quad = -\eth\dot{\bar{\sigma}}_{\mathrm{B}} + 2\dot{\bar{\sigma}}_{\mathrm{B}} H_2, \end{aligned} \tag{92}$$

would be our sought for equation for the determination of the characteristic surfaces and therefore the conformal metric of the Einstein space–time.

The equation, however, can be manipulated, with the help of the leading terms (but not explicitly) of the remaining $\Lambda(H)$-relations, into a simpler and more attractive equation that is referred to as the light cone cut equation (LCCE) whose solution directly yields the light cone cut function.

From the solution to Eq. (91) (using the explicit form for X and Y) in the form

$$H_1 = \tfrac{1}{2}\Lambda_1 + \mathrm{O}(\Lambda^2) \tag{93}$$

we have from substitution into Eq. (92)

$$\bar{\eth}^3 \Lambda_1 = -4\eth\dot{\bar{\sigma}}_{\mathrm{B}} + \text{higher-order terms}, \tag{94}$$

where σ_{B} is free data given on $\mathcal{I}^+$. Manipulating this and using Eq. (86) of the $\Lambda(H)$-relations we obtain, as

$$\bar{\eth}^2 \Lambda_0 = \eth^2\dot{\bar{\sigma}}_{\mathrm{B}} + \bar{\eth}^2\dot{\sigma}_{\mathrm{B}} + \text{higher-order terms}, \tag{95}$$

or, from the definition $\eth^2 Z_{,0} = \Lambda_0$,

$$\bar{\eth}^2\eth^2 Z = \eth^2\bar{\sigma}_{\mathrm{B}} + \bar{\eth}^2\sigma_{\mathrm{B}} + \text{higher-order terms} \tag{96}$$

as our final equation for the light cone cut function Z. Although the higher-order terms have not been worked out explicitly, they can be obtained via a perturbation scheme (see Section 4). The solutions to Eq. (96) yield the conformal structure of asymptotically flat vacuum space–time.

In the above analysis it was assumed that the solution to the equation for H_1, namely,

$$\frac{\partial^2 H_1}{\partial R^2} + X\frac{\partial H_1}{\partial R} + YH_1 = 0 \tag{97}$$

was known. Since it is a linear second-order o.d.e. solutions for arbitrary $\Lambda(R)$ must exist, though it is not clear—nor is it likely—that we will be able to solve it explicitly. It will however, always be of the form

$$H_1 = \tfrac{1}{2}\Lambda_1 + \mathrm{O}(\Lambda^2) \tag{98}$$

with the higher-order terms computable perturbatively.

Since the LCCE (96) determines the conformal structure of vacuum space–times, the only remaining quantity to be determined is the conformal factor needed to convert the conformal metric into a vacuum metric. In the θ^i coordinate system the relation between the conformal metric g and the full metric $\hat{g}$ is given by

$$\hat{g}^{ij} = \Omega^2 g^{ij}. \tag{99}$$

It can be shown from Eq. (62) [12] that the conformal factor can be expressed as a simple function of H_1 and $\overline{H}_1$, namely,

$$\Omega^2 = (1 + h_1)(1 + \bar{h}_1) + H_1\overline{H}_1. \tag{100}$$

(As an aside we remark that from Eq. (97), it is possible to show that the conformal factor (100) satisfies the equation

$$\frac{\mathrm{d}\Omega}{\mathrm{d}R^2} = Q(\Lambda)\Omega, \tag{101}$$

where $Q = Q(\Lambda)$ is a simple known function of Λ_1. The above equation is known in the literature [13] as the Einstein bundle equation.)

4. An iterative scheme

Much of the understanding of the structure and meaning of our final equations has come from the application of an iterative scheme to the formal theory and looking at the leading behavior. In this section, we present the results obtained in linear order.

We expand all quantities in powers of a small parameter ϵ which measures the deviation from flatness. From the assumption that ϵ enters as a multiplicative factor of the Bondi shear, i.e., via $\epsilon\sigma_{\rm B}$, it becomes clear that the expansions have the form

$$Z = \overset{(0)}{Z} + \epsilon \overset{(1)}{Z} + \epsilon^2 \overset{(2)}{Z} + \cdots, \tag{102}$$

$$e^i_a = \overset{(0)i}{e}{}_a + \epsilon \overset{(1)i}{e}{}_a + \epsilon^2 \overset{(2)i}{e}{}_a + \cdots, \tag{103}$$

$$\Lambda = \epsilon \overset{(1)}{\Lambda} + \epsilon^2 \overset{(2)}{\Lambda} + \epsilon^3 \overset{(3)}{\Lambda} + \cdots, \tag{104}$$

$$H = \epsilon \overset{(1)}{H} + \epsilon^2 \overset{(2)}{H} + \epsilon^3 \overset{(3)}{H} + \cdots, \tag{105}$$

$$h = \epsilon^2 \overset{(2)}{h} + \epsilon^3 \overset{(3)}{h} + \cdots \tag{106}$$

There is no zeroth order contribution to Λ or the holonomy operator since these quantities are zero for Minkowski space. Furthermore, a direct calculation (using Eqs. (87), (89) and (93)) shows that h begins at second order.

Since Λ starts with order one in the perturbation expansion, it follows from the relationship between Z and Λ that $\overset{(0)}{Z}$ satisfies

$$\eth^2 \overset{(0)}{Z}_a = 0, \tag{107}$$

whose solution is the Minkowski space light cone cut function

$$\overset{(0)}{Z} = x^a \overset{(0)}{\ell}_a, \tag{108}$$

where

$$\overset{(0)}{\ell}_a (\zeta, \bar{\zeta}) = \frac{1}{\sqrt{2}P}(1 + \zeta\bar{\zeta}, \zeta + \bar{\zeta}, i(\zeta - \bar{\zeta}), -1 + \zeta\bar{\zeta}), \quad P = (1 + \zeta\bar{\zeta}). \tag{109}$$

4.1. Linearized gravity

We begin the linearization with the three holonomy Bianchi identities, which are of first order:

$$\overset{(1)}{H}_2 = \tfrac{1}{2}\bar{\eth} \overset{(1)}{H}_1, \tag{110}$$

$$\overset{(1)}{H}_3 = -\bar{\eth}\,\overset{(1)}{H}_2\,, \tag{111}$$

$$\bar{\eth}\,\overset{(1)}{H}_3 = \eth\dot{\bar{\sigma}}_{\mathrm{B}}, \tag{112}$$

where the h's have disappeared since they have no first-order contribution. The next step in the process of elimination would naturally be to solve the holonomy field equations obtaining the H's in terms of the Λ. However, to first (and second order) we can circumvent this step and consider the first (Λ, H)-equations directly. It leads to an expression for H_1, which, with Eqs. (110) and (111), yields the first-order H_α that automatically satisfy the field Eqs. (91) to first order. (For third and higher order we would have to solve the field equations *before* using the $\Lambda(H)$-relations for the simplification procedure.)

We thus have

$$\overset{(1)}{H}_1 = \tfrac{1}{2}\,\overset{(1)}{\Lambda}_1\,, \tag{113}$$

$$\overset{(1)}{H}_2 = \tfrac{1}{4}\bar{\eth}\,\overset{(1)}{\Lambda}_1\,, \tag{114}$$

$$\overset{(1)}{H}_3 = -\tfrac{1}{4}\bar{\eth}^2\,\overset{(1)}{\Lambda}_1\,. \tag{115}$$

Our final equation for the cut function is the linearized third Bianchi identity, which, after substituting for H_3, is

$$\bar{\eth}^3\Lambda_1 = -4\eth\dot{\bar{\sigma}}_{\mathrm{B}}. \tag{116}$$

(Although we did not have to solve the holonomy field equations, it is a straightforward but slightly tedious process to verify that the expressions (113)–(115) for the H's do satisfy the field equations. A minor subtlety in the calculation is that the linearization of the field equations yields identically vanishing expressions, so that one must look at the equations at their first *non-vanishing* order. Performing the calculation, nevertheless, serves as a consistency check between the different sets of equations.)

At this point we have used all the equations except the three remaining $\Lambda(H)$-relations,

$$\Lambda_- = \tfrac{1}{2}\eth\Lambda_1, \tag{117}$$

$$\Lambda_+ = -\tfrac{1}{2}\bar{\eth}\Lambda_1, \tag{118}$$

$$\Lambda_0 = \dot{\sigma}_{\mathrm{B}} + \tfrac{1}{2}\Lambda_1 - \tfrac{1}{4}\eth\bar{\eth}\Lambda_1. \tag{119}$$

These relations can be used to write (116) in the more symmetric form as the linearized version of the light cone cut equation:

$$\bar{\eth}^2\eth^2 Z = \eth^2\bar{\sigma}_{\mathrm{B}}(Z) + \bar{\eth}^2\sigma_{\mathrm{B}}(Z), \tag{120}$$

which is the light cone cut equation accurate to first order. In other words, this equation is equivalent to the linearized conformal Einstein equations. (The above equation for linear theory has been derived from the vanishing of the Bach tensor, by Lionel Mason [14].)

In linear theory (see Eq. (100)) the conformal factor equals 1, and therefore we have in this approximation we have not only the conformal Einstein equations but the Einstein equations themselves.

Eq. (120), is to be thought of as an equation for Z, whose solution can be written as a sphere integral using the Green function of the operator $\bar{\eth}^2\eth^2$. See Appendix C of [12] for a derivation of the Green function $\mathcal{G}$ of this operator. In other words we can write

$$Z = \int (\mathcal{G}\eth^2\bar{\sigma}_{\mathrm{B}}(Z) + \bar{\eth}^2\sigma_{\mathrm{B}}(Z)\,\mathrm{d}R, \tag{121}$$

with $\mathrm{d}\,S_\eta$ as the sphere volume element in $(\eta, \bar{\eta})$ coordinates, as the general solution to the asymptotically flat linearized vacuum Einstein equations. In principle, though we have not yet done so in practice, the higher-order terms could be calculated successively by similar integrals but with the integrands depending on terms to lower order in the perturbation scheme.

5. Summary and conclusions

We have obtained three sets of coupled equations, the holonomy Bianchi identities, the $\Lambda(H)$-relations and the "field equations" for the holonomy operator H and the light cone cut function Z which are equivalent to the full vacuum Einstein equations. These equations, which already have built into them the free choice of Bondi data $\sigma_{\mathrm{B}}(u, \zeta, \bar{\zeta})$, can be manipulated and simplified (in structure) to one (complicated) equation, the LCCE, and a simple one for the conformal factor. On analysis, they yield perturbatively, a D'Adhemar-like formulation of general relativity.

On a negative side these equations are quite unusual and are based on unfamiliar ideas and variables, and unfortunately are quite complicated. They nevertheless have some features of considerable attractiveness: new insights often can be gained from the use of new variables; the perturbative solutions, from given data, are essentially unique; as the perturbation calculation proceeds to higher order to formalism yields the corrected light cone structure from the preceding order, in contrast to the usual perturbation theory which uses, at all orders, the Minkowski light cone structure.

As a final comment, we remark that the work reported on here is the direct antecedent of another approach to General Relativity (the Null Surface Theory of GR, to be reported elsewhere) that is based solely on families of characteristic surfaces as the basic variable of the theory—without any mention of holonomies. The new view, which has certain similarities to the present work and will probably yield considerable simplifications over it, could not have been developed without the current approach.

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